Portfolio Management
Mean Variance Analysis Theory

- Portfolio Management is a very widely studied subject and lots of quantitative and qualitative research has been done in this subject
- Mean Variance portfolio theory provides the foundation of modern portfolio theory and is very useful in examining the role of risk and return in portfolio selection
  - Basic idea for mean variance theory is that value of investment opportunity can be measured in terms of mean return and variance of return
- Assumptions for Mean Variance theory
  - All investors are risk averse, they prefer less risk for the given level of expected return
    - This does not mean that all investors have same tolerance or risk, investors are differ in level of risk, but Risk Averse investor prefer as little risk as possible for given level of expected return
  - Expected return of all assets are known
  - The Variance and covariance of all assets are known
  - Investors needs to know only expected return, variance and covariance of returns to determine optimal portfolios
  - There are no transaction cost or tax
Calculation of Expected Return and Standard Deviation of return of a portfolio

• Expected return of a single security is measured by multiplying the probability of achieving a return with the quantum of the return. The various return are summed to arrive at the security return.

\[ E(R) = P_1R_1 + P_2R_2 + P_3R_3 + \ldots + P_nR_n \]

– Where \( P_1, P_2, \ldots, P_n \) shows the probability of getting expected return

• For a portfolio consisting of two assets, the portfolio’s mean return is calculated as –

\[ E(R_p) = w_1E(R_1) + w_2E(R_2) \]

– Where \( E(P_1) \) and \( E(P_2) \) are expected returns on assets consisting portfolio

• Portfolio variance for the portfolio consisting of two assets are as follows –

\[ \sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \]

– Where \( \sigma_1, \sigma_2 \) and \( \sigma_p \) are standard deviations and \( \rho_{12} \) is correlation of return between two assets

• Thus portfolio standard deviation consisting of two assets is -

\[ \sigma_p = (w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2)^{1/2} \]
Calculation of Expected Return and Standard Deviation of return of a portfolio - Example

Expected Annual Return for Stock A = 11.35%
Expected Annual Return for Stock B = 9.52%
Annual Std. Dev. of Return for Stock A = 15.38%
Annual Std. Dev. of Return for Stock B = 13.40%
Correlation of return of Stock A and Stock B = 0.0006

• Assume that both the stocks in a portfolio has equal weights of 50%
  – Thus portfolio mean return can be calculated as –

    Mean Return = 0.5 x 11.35% + 0.5x 9.52% = 10.435%

  – Portfolio standard deviation can be calculated as –

    Standard Deviation = [0.5^2 x (15.38%)^2 + 0.5^2 x (13.40%)^2 + 2 x 0.5 x 0.5 x 0.0006 x 15.38% x 13.40%]^{1/2}
    = 10.2023%

  – Similarly expected return, variance and standard deviation of return of the portfolio for different weight age
    of the asset in a portfolio can be calculated
  – Using various combinations of assets in a portfolio and calculating expected return and variance of the
    portfolio

    • All this means and variance of the portfolio can be plotted on the axis, graph of which is called the portfolio
      possibility curve for a given portfolio
Minimum Variance Frontier

- Portfolio Possibility curve for a portfolio –

- This portfolio possibility curve is also called minimum variance frontier, because it shows the minimum that can be achieved for a given level of expected return
  - More useful concept than portfolio possibility curve because it can be applied to portfolio with any number of assets
  - All points lying above minimum variance frontier are called minimum variance portfolios
  - Minimum variance portfolio is the point with the lowest possible risk for a given level of expected return, on a horizontal line on the axis with portfolio mean plotted on the y-axis and portfolio’s variance plotted on x-axis
  - If we move right from a point on the minimum variance frontier, we can get portfolio with same return but with more risk
Global Minimum Variance Portfolio

- Minimum variance frontier shows all possible minimum variance portfolios
  - Still there are points, lying on the minimum frontier which are dominated by other points having same mean return but having lesser variance than the former

  - For example: In above figure point A is having same variance as point B, but having higher return than point B
  - The point lying on the minimum variance frontier having lowest variance is called Global Minimum Variance Portfolio

Remember, only the part marked on red is relevant...why?
**Efficient Frontier**

- Portion of the minimum variance frontier beginning with the Global Minimum Variance Portfolio is called Efficient Frontier

![Efficient Frontier Diagram]

- Efficient frontier simplifies selection process as it removes all inefficient portfolios lying on minimum variance frontier
  - Portfolio lying on the efficient frontier offers the maximum return for given level of risk
  - Investors making portfolio choices based on mean return and variance of return should limit their choices to portfolios lying on efficient frontier
Portfolio Diversification – Two Assets Case

- Risk – Return tradeoff for a portfolio not only depend on expected return and variance of return of the portfolio, it also depends on the correlation between the asset held in portfolio.
Portfolio Diversification – Two Assets Case

- Shape of the minimum variance frontier and diversification benefits
  - Whatever correlation between the assets in the portfolio is, the end points of the corresponding minimum variance frontier will be the same
  - When the correlation between assets in a portfolio is +1, the minimum variance frontier is an upward sloping line
    - No diversification can be achieved in this case
  - When correlation between two assets is 0.5 it is bowed towards left with the same end points
  - When the correlation among assets in a portfolio is 0, it is more bowed towards left with the same end points
  - When the correlation between the assets is -1, the minimum variance frontier has two linear segments which joint at a point having zero standard deviation
    - 100% diversification can be achieved in this case

- Lower the correlation, higher the diversification benefits can be achieved
Portfolio Diversification – Three Assets Case

• For a portfolio having three assets –

\[
\text{Portfolio Mean Return} = w_1 E(R_1) + w_2 E(R_2) + w_3 E(R_3)
\]

• Portfolio variance can be calculated as –

\[
\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2 + 2w_1 w_2 \rho_{12} \sigma_1 \sigma_2 + 2w_2 w_3 \rho_{23} \sigma_2 \sigma_3 + 2w_3 w_1 \rho_{13} \sigma_3 \sigma_1
\]

– Where \( \rho_{12}, \rho_{23} \) and \( \rho_{13} \) shows the covariance of return between assets

• After calculating the portfolio mean return and portfolio variance for portfolio having three assets, minimum variance frontier can be plotted for a portfolio using a spreadsheet program “optimizer”.

\[
\text{Portfolio Mean Return} = \sum_{i=1}^{3} w_i E(R_i)
\]

\[
\sigma_p^2 = \sum_{i=1}^{3} w_i^2 \sigma_i^2 + \sum_{i=1}^{2} \sum_{j=i+1}^{3} w_i w_j \rho_{ij} \sigma_i \sigma_j
\]
Portfolio Diversification – Three Assets Case

- The minimum variance frontier for three assets case and two assets case is –

![Graph showing Minimum Variance Frontier for Three and Two Assets Case]

- Lower the diversification among assets in a portfolio, the greater the diversification benefits can be achieved
Determining Minimum Variance Frontier for a portfolio having many assets -

• For a portfolio having n assets –
  – Expected return on the portfolio is –

\[
E(R_p) = \sum_{j=1}^{n} w_j E(R_j)
\]

  – Where \( w_j \) is the weight of the assets in a portfolio, \( \sum w_j = 1 \)

  – Variance of the portfolio is –

\[
\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(R_i, R_j)
\]

  – To determine minimum variance frontier for a portfolio having n assets, first the minimum \( R_{\text{min}} \) and maximum \( R_{\text{max}} \) expected returns possible with the set of assets should be determined

  – Then minimum variance frontier can be determined using –

\[
\text{Minimize } \sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(R_i, R_j)
\]

  – Where \( E(R_p) = \sum w_j E(R_j) \) and \( \sum w_j = 1 \)
Diversification and Portfolio Size

- How many different stocks must an investor hold in order to have a well diversified portfolio?
- How does covariance interact with portfolio size in determining a portfolio’s risk?

- Considering portfolio of \( n \) stocks having equal weightage in a portfolio (The weight \( w_i = \frac{1}{n} \)), and putting this in equation –

\[
\sigma_p^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} w_i w_j \text{Cov}(R_i, R_j)
\]

- Variance of return for a portfolio is –

\[
\sigma_p^2 = \sum_{i=1}^{n} w_i w_j \text{Cov}(R_i, R_j) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov}(R_i, R_j)
\]
Diversification and Portfolio Size

• Assuming average variance of return across all stocks as $\overline{\sigma^2}$ and the average covariance between all pairs of two stocks as $\overline{\text{Cov.}}$, then equation becomes –

$$\sigma_p^2 = \frac{1}{n} \overline{\sigma^2} + \frac{n-1}{n} \overline{\text{Cov.}}.$$

• As the number of stock n increases
  – $1/n$ tends to zero
  – $(n-1)/n$ tends to 1
  – Hence, portfolio becomes approximately equal to average covariance (as average covariance across stocks to the portfolio variance stays nonzero)
  – In large portfolios, average covariance (Capturing how assets move together) becomes more important than average individual risk or variance

• Assuming all stocks have same standard deviation of returns –

$$\sigma_p^2 = \sigma^2 \left( \frac{1-\rho}{n} + \rho \right)$$

– Where $\rho$ is the average correlation between stocks in a portfolio
• Increasing number of stocks in a portfolio having lower correlation among each other results in a reduction of portfolio variance
• One common belief among investors that almost all benefits of diversification can be achieved with a portfolio with 30 stocks

• Important Conclusion:
  – Higher correlation among assets in a portfolio results in a higher level of minimum variance achievable for a portfolio
    • In order to achieve certain percentage (e.g. 150%) of minimum variance, less number of stocks needed
  – Lower correlation among assets in a portfolio results in a lower level of minimum variance achievable for a portfolio
    • In order to achieve certain percentage of minimum variance, more number of stock needed for a portfolio
The Capital Allocation Line

- A risk free asset’s standard deviation of return is zero as there is no risk of default and it’s correlation of return with any risky securities is also zero

- Capital Allocation Line (CAL):
  - It can be created by combining the risk free assets with the portfolio consisting of risky assets
  - It describes the expected results of the investor’s decision on how to optimally allocate capital among risky and risk free assets
Important principles concerning the risk return trade off in a portfolio containing a risk free asset:

- **CAL** represents the best risk return tradeoff achievable when combining risk free assets with a risky portfolio.
- The **CAL** has y-intercept equal to risk free rate.
- The **CAL** is tangent to efficient frontier of risky assets.

Assume an investor investing \( w_t \) portion of his portfolio weight in risky assets and rest in a risk free asset then the expected return on the entire portfolio is:

\[
E(R_p) = (1 - w_t) R_F + w_t E(R_t)
\]

As risk free assets have zero correlation with any risky assets, including a risk free assets in portfolio doesn’t add any change in risk the risky assets have thus –

\[
\sigma_p = W_t \times \sigma_{R_t}
\]

\[
W_t = \frac{\sigma_p}{\sigma_{R_t}}
\]
• Substituting $\sigma_p / \sigma_{Rt}$ for $w_t$ in a expected return equation –

$$E(R_p) = R_F + \left( \frac{E(R_t) - R_F}{\sigma_t} \right) \times \sigma_p$$

– The term $[E(R_t) - R_F] / \sigma_t$ is the return that investor demands in order to take on an extra unit of risk
The Capital Market Line

• When investors have identical expectations about the mean returns, variance of returns and
  correlation of risky assets -
  – The CAL for all investors is the same and is known as capital market line (CML)
  – This tangency portfolio must be a portfolio containing all assets in proportion to their market value
    weights, hence market portfolio of risky assets
  – CML is a capital allocation line with market portfolio as tangency portfolio

• Equation for capital allocation line is –

\[
E(R_p) = R_F + \left( \frac{E(R_M) - R_F}{\sigma_M} \right) \times \sigma_p
\]

  – Where \( E(R_M) \) is the expected return on market portfolio and \( \sigma_p \) is standard deviation of return on market
    portfolio
  – The term, \( \frac{E(R_M) - R_F}{\sigma_M} \) is also the market price of risk and it indicates the market risk premium for
    each unit of risk